Change Analysis of Default Correlation and Dynamic Pricing of Collateralized Debt Obligations*

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Abstract

In this paper we use dynamic copulas method to price a CDO. We apply GOF test and binary segmentation procedure to detect the change of copula function. According to the result of the change point, we divide the time series into nine stages. In each stage, we use the best copula function to describe the default correlation. Our empirical results show that in different time period, the best copula fitting to data set is not static, thus the expected loss and fair spread are different for each stages. This explains why investors of CDO suffered so much when financial crisis happened. We also give a comparison among static copula model, dynamic Gaussian copula model and dynamic copula model, in the end we find dynamic copula model not only include the tail dependence in default structure, but also provide a more insensitive way to price CDO.

Keywords: CDO pricing, Default correlation, Dynamic copula, Goodness-of-fit test, Change point analysis

1 Introduction

In recent years credit derivatives have become a main tool to transfer and hedge credit risk. Among these credit derivatives, Collateralized Debt Obligation(CDO), which is a kind of security consisting of a pool of assets (e.g. bonds, loans, credit default swaps and commercial mortgages), becomes the most rapidly developed item in derivative markets. CDOs transfer the risk of reference entities to investors who are willing to take these risks and get benefit from them. A CDO is decomposed into several tranches, each of which has an attachment percentage and a detachment percentage. Investors in each tranche start to lose their principal when the cumulative percentage loss of the portfolio reaches their attachment percentage, and when the cumulative percentage loss gets to the detachment percentage, the investors in the tranche lose all their principals but no more beyond that. In other words, CDO provides a protection for investors under this situation.

There are two major methods to price credit derivatives: structural approach initiated by Merton (1974) and reduced-form approach proposed by Jarrow and Turnbull (1995). A critical point to the pricing of credit derivatives is how to characterize the default correlation which becomes even more difficult for CDO pricing. In order to price CDO tranches, it is essentially to find out the marginal distribution of all reference entities and their joint distribution. For the problem of joint distribution, copula method has been widely used because of its convenience to characterize the dependence structure. Li (2000) suggested to use Gaussian copula in credit risk modeling to estimate the default correlation and this method had been widely used in Wall Street. However, when using Gaussian copula to model the default correlation, one can detect the smile effect which is explained to be lack of tail dependence and underpricing the upper tranches of a CDO. To fix this deficiency, many authors proposed different approaches to bring more tail dependence into the model. Glasserman and Suchintabandid (2007) used correlation expansions for CDO pricing but still under the approach of Gaussian copula. Hull and White (2004) proposed the double t distribution; and Anderson and Sidenius (2005) introduced Marshall-Olkin copula which are more capable to capture the tail dependence. Xu (2006) used the mixture copula model of multi Gaussian distributions, while Wang et al (2006) used double mixture of student t and Gaussian copula. Yang and Qin (2009) used mixture of NIG and Gaussian copula to price CDO tranches. They all found that mixture copula fitted better than the single one. Besides these, Grane and Hoek (2008) suggested to use distortions of copulas to produce a heavy tailed portfolio loss distribution and price synthetic CDOs.

All of the methods mentioned above are based on a common assumption of static dependence structure. However, financial data often covers a long time period, so the dependence structure tends to change instead of keeping static. As we all know, the recent financial crisis is closely related to the characterization of default correlation and pricing of CDO. This crisis brought a critical issue to related communities: how to price those multi-name credit derivatives, such as CDO under different situations. In this paper we will use dynamic copulas to analyze default correlation and pricing CDO.

Dias and Embrechts (2009) use time varying copula models to analyze the dependence structure between foreign exchange returns on USD/DEM and USD/JPY spot rates across six different time frequencies from one hour to one day. Their work shows an improvement to the time invariant copula model. Zhang and Guegan (2008) employ GARCH process together with time-varying copula to price bivariate options and the result shows that dynamic copulas together with time-varying parameters offer a better alternative to any static model for dependence structure. Zhang (2008) apply dynamic copula method to measure the dynamic dependence structure of financial data and calculate the dynamic VaR of S&P500 and Nasdaq Index.

There are two types of changes for the default correlation: one is the change of copula parameters while the copula family keeps the same; another one is the change of copula family. In this paper, we use dynamic copula method to describe the time-varying dependence structure between reference entities and propose a dynamic pricing model for CDO. According to the binary segmentation procedure, we divide the time sample into different stages. In each stage, we use maximum likelihood principle to choose the copula that best fitting the date. In our empirical work, we find that the dependent structure of financial data changes from time to time, especially when a financial event busted out. So in different time periods, we use different copulas, each of which is the best one for that period, to describe the dependent structure. Then we get the joint distribution of all reference entities in CDO and then price each tranche of it.

The rest of this paper is organized as follows. In Section 2 we introduce the dynamic copula model and the methods will be used in our empirical study; Section 3 describes the methods to calculating the expected loss and fair spread of a CDO; An empirical study is presented in Section 4 to demonstrate the implementation of our method; Section 5 concludes the paper.

2 Dynamic Copula Model

In this section, we will first simply introduce the dynamic copula model, then we will introduce Goodness-of-fit test which will be used to test copula's changes in our empirical work. At last, we will give a detailed analysis for the changes of copulas.

2.1 Copulas and Dynamic Copulas

An n-dimensional copula is a function $C : [0,1]^n \to [0,1]$ with the following properties:

(i)(Grounded) $C(u) = C(u_1, u_2, \ldots, u_n) = 0$ when at least one coordinate u_i is equal to zero.

(ii)If all coordinate of $\mathbf{u}$ are equal to one except u_k then $C(u) = C(u_1, u_2, \ldots, u_n) = u_k,\ k = 1, 2, \ldots, n$.

(iii)(Increasing) C is n-increasing.

According to copula's definition, Sklar (1959) indicate that a copula function is essentially the joint distribution of n random variables: Let F be the joint distribution of n random variables with marginal distributions $F_1, F_2, \ldots, F_n$, then there exists a copula function C satisfying:

$$F(x_1, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{1}$$

If $F_1, F_2, \ldots, F_n$ is continuous, then C is unique. There are several typical copulas, such as Gaussian copula, Student t copula and Archimedean copulas. For details please see Nelsen (1999) or Cherubini et al (2004).

As introduced in introduction, the dynamic copula model is divided into two parts: the changes of parameters and the changes of copula itself. Let $X_{1t}, X_{2t}, ..., X_{nt}$ be an n-dimensional stochastic process, then a dynamic copula can be described as follows:

$$
\begin{aligned}
&X_{1t} \sim F_{1t}(x_{1t}; \gamma_{11}), \quad t = 1, 2, ..., s_{11}, \\
&X_{1t} \sim F_{1t}(x_{1t}; \gamma_{12}), \quad t = s_{11} + 1, s_{11} + 2, ..., s_{12}, \\
&\vdots \\
&X_{1t} \sim F_{1t}(x_{1t}; \gamma_{1k_1}), \quad t = s_{1,k_1-1} + 1, s_{1,k_1-1} + 2, ..., T, \\
&X_{2t} \sim F_{2t}(x_{2t}; \gamma_{21}), \quad t = 1, 2, ..., s_{21}, \\
&X_{2t} \sim F_{2t}(x_{2t}; \gamma_{22}), \quad t = s_{21} + 1, s_{21} + 2, ..., s_{22}, \\
&\vdots \\
&X_{2t} \sim F_{2t}(x_{2t}; \gamma_{2k_2}), \quad t = s_{2,k_2-1} + 1, s_{2,k_2-1} + 2, ..., T, \\
&\vdots \\
&X_{nt} \sim F_{nt}(x_{nt}; \gamma_{n1}), \quad t = 1, 2, ..., s_{n1}, \\
&X_{nt} \sim F_{nt}(x_{nt}; \gamma_{n2}), \quad t = s_{n1} + 1, s_{n1} + 2, ..., s_{n2}, \\
&\vdots \\
&X_{nt} \sim F_{nt}(x_{nt}; \gamma_{nk_n}), \quad t = s_{n,k_n-1} + 1, s_{n,k_n-1} + 2, ..., T, \\
&(X_{1t}, X_{2t}, ..., X_{nt}) \sim C_1(F_{1t}(x_{1t}), F_{2t}(x_{2t}), ..., F_{nt}(x_{nt}); \theta_1), \quad t = 1, 2, ..., s_1, \\
&(X_{1t}, X_{2t}, ..., X_{nt}) \sim C_2(F_{1t}(x_{1t}), F_{2t}(x_{2t}), ..., F_{nt}(x_{nt}); \theta_2), \quad t = s_1 + 1, s_1 + 2, ..., s_2, \\
&\vdots \\
&(X_{1t}, X_{2t}, ..., X_{nt}) \sim C_k(F_{1t}(x_{1t}), F_{2t}(x_{2t}), ..., F_{nt}(x_{nt}); \theta_k), \quad t = s_k + 1, s_k + 2, ..., T.
\end{aligned}
$$

where $F_{it}(\bullet;\gamma)$ is the ith marginal distribution with parameters γ; $C_k(...,\theta_k)$ is the n-dimensional copula function with parameters θ_k at a specific time; s is the change point for marginal distribution and copula function. More specifically, s_{ij} is the jth change point of the ith marginal distribution and s_m is the mth change point in copula function. T is the size of sample.

2.1.1 Goodness-of-Fit Test

We will use goodness-of-fit to test whether an empirical copula is stable or change at a certain time.

Suppose that the unknown copula C belongs to a parametric copula family: $C_0 = C_\theta : \theta \in \Omega$, where Ω is an open sub-set of R^q for some integer $q \geq 1$. A GOF test is to distinguish between two hypotheses:

$$H_0 : C \in C_0 \quad or \quad H_1 : C \notin C_0.$$

To implement the test, many procedures have been proposed. Among these literatures Genest et al (2009) suggests that, based on large scale simulations, powerful GOF tests can be obtained from the process:

$$\overline{C}_n(u) = \sqrt{n} C_n(u) - C_{\theta_n}(u), \quad u \in [0,1]^d \tag{2}$$

where C_n is the empirical copula of samples $X_1, X_2, ..., X_n$:

$$C_n(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\hat{U}_i \leq u\}}, \quad u \in [0,1]^d \tag{3}$$

and C_{θ_n} is an estimator of θ calculated by the maximum pseudo-likelihood approach proposed in Genest et al (2009), Shih and Louis (1995). To obtain approximate p-values for tests based on statistics derived from the GOF test such as Cramer-von Mises statistic, Genest el al. (2009) applied the parametric bootstrap approach and obtained good results. However, this procedure has an extremely high computational cost when the size of sample becomes large, which is regarded as an obstacle to its application. An alternative approach proposed by Ivan Kojadinovic and JunYan (2009) is to use a multiplier procedure. Because of its convenience and efficiency, we decide to use this procedure in our empirical work. For detailed analysis of multiplier procedure please see corresponding literatures.

2.1.2 Dynamic Copula Analysis

If the result of test suggests that the dependence structure may change, we have to detect the change points. In this paper, we employ the binary segmentation procedure proposed by Vostrikova (1981) to find out the change points.

The binary segmentation procedure is as follows: we first choose the best copula according to the AIC criterion on the whole sample, then divide the whole sample into two sub-samples, and choose the best copulas on these two sub-samples respectively. If the two best copulas are different from the copula on the whole sample, we continue this segmentation procedure, and divide each sub-sample into two parts again, and do the same work as in the previous step. At last, the procedure ends when all the best copulas on the sub-samples have been adjusted. From this method, we can get all the change points for our model.

3 CDO Pricing Model

In this section, we give the pricing model for a standard CDO. Firstly, we consider the issue of default probability. Let τ be default time, then the probability that an obligor defaults within time t is $F(t) = P(\tau \leq t)$. This function can also be written as:

$$F(t) = 1 - \exp(-\int_0^t \lambda(u)du) \tag{4}$$

where $\lambda(t)$ is the default intensity or hazard rate function. In real world hazard rate can't be observed at all time, so when pricing credit derivatives we always assume that $\lambda(t)$ is deterministic and can be obtained from the following equation (See Cherubini et al (2004)):

$$\lambda(t) = \frac{S_t}{1 - R} \tag{5}$$

where S_t is the credit spread and can be observed in real markets; R is the recovery rate. In this paper, the ith obligor's default time can be described to be:

$$\tau_i := \inf\{t : \int_0^t \lambda_i(u)du \geq E_i\} \tag{6}$$

where E_i are exponential r.v.s of parameter $\mathbf{1}$ and independent of the intensity process.

To obtain the joint distribution of default time, we apply copula function into our model. From the definition of copula function, we can get the joint default distribution function as follows:

$$Pr(\tau_1 \leq t_1, \tau_2 \leq t_2, ..., \tau_n \leq t_n) = C(F_1(t_1), F_2(t_2), ..., F_n(t_n)) \tag{7}$$

From above equation we can obtain correlated default times, and then price a CDO. For simplicity, we assume that default time, risk-free interest rates and recovery rates are independent. The CDO includes n reference obligors, each of which has a nominal amount A_i and recovery rate R_i with $i = 1, 2, ..., n$. If the ith obligor defaults, then the loss is equal to $L_i = (1 - R_i)A_i$. Let $N_i(t) = \mathbf{1}_{\{\tau_i < t\}}$ be the counting process which jumps from 0 to 1 when the ith obligor default. Then we can obtain the cumulative loss of the collateral portfolio at time t:

$$L(t) = \sum_{i=1}^{n} L_i N_i(t) \tag{8}$$

Suppose a CDO tranche has an attachment point C and a detachment point D, satisfying $0 \leq C \leq D \leq \sum_i^n A_i$, then the cumulative losses on a given tranche M_t is:

$$M(t) = [L(t) - C]\mathbf{1}_{\{C,D\}}[L(t) + (D - C)]\mathbf{1}_{\{D, \sum_{i=1}^n A_i\}}(L(t)) \tag{9}$$

When $C = 0$, we call it the equity tranche; when $C > 0$ and $D < \sum_i^n A_i$, we consider the mezzanine tranches, and when $M = \sum_i^n D_i$, we refer to senior or superior tranches. The investor in a tranche is equivalent to sell a default protection to sponsors and get some premium or spread for undertaking the risk of CDO. To obtain a fair spread, we calculate the default payment leg (DL) and premium leg (PL) at $t = 0$.

Let $B(0,t)$ be the discount factor at time t, and T be the maturity of CDO, then we can write the default leg of the given tranche as:

$$DL = E^Q[\int_0^T B(0,t)dM(t)] \tag{10}$$

Assume t_i denotes the premium payment date, $\Delta_{i-1,i} = t_i - t_{i-1}$ denotes the tenor between successive premium payment dates, and W is the fair spread, then the premium leg is:

$$\begin{aligned} PL &= E^Q[\sum_{i=1}^m \Delta_{i-1,i} \cdot W \cdot B(0,t_i) \cdot [D-C] \cdot \mathbf{1}_{\{L(t) \le C\}} \\ &\quad + \sum_{i=1}^m \Delta_{i-1,i} \cdot W \cdot B(0,t_i) \cdot [D-C] \cdot \mathbf{1}_{\{C \le L(t) \le D\}}] \end{aligned} \tag{11}$$

In a continuous time setting, we can express the discounted value at time 0 of the premium leg as:

$$W \cdot E^Q[\int_0^T B(0,t)g(L(t))dt] \tag{12}$$

where $g(L(t)) = \min\{\max[D - L(t), 0], D - C\}$. Then the fair spread W can be obtained:

$$W = \frac{E^Q[\int_0^T B(0,t)dS(t)]}{E^Q[\int_0^T B(0,t)g(L(t))dt]} \tag{13}$$

In the following section we use Monte Carlo simulation to calculate the fair spread.

4 Empirical Study

4.1 Data

In this section, we apply the dynamic copula model to price a CDO. First, we assume that a CDO is composed of 6 obligations that come from the companies belonging to Chinese stock market, and each single item has the principal of 10 million dollars. Hence the collateral portfolio has a nominal amount equal to 60 million dollars. For simplicity, suppose that the CDO has a maturity of 1 year and the risk-free interest rate is equal to 10% per year. For simplicity, the credit spreads and corresponding recovery rates of all obligations are set to 100 bps and 0.4 separately. The profit and loss structures of the CDO are reported in Table 1.

Table 1: Tranches and the loss undertaken.

Tranche	Size(%)	Subordination(%)	Loss Undertaken(%)
A	10	[0,10]	[0,10]*60m
B	10	[10,20]	[10,20]*60m
C	15	[20,35]	[20,35]*60m
D	65	[35,100]	[35,100]*60m

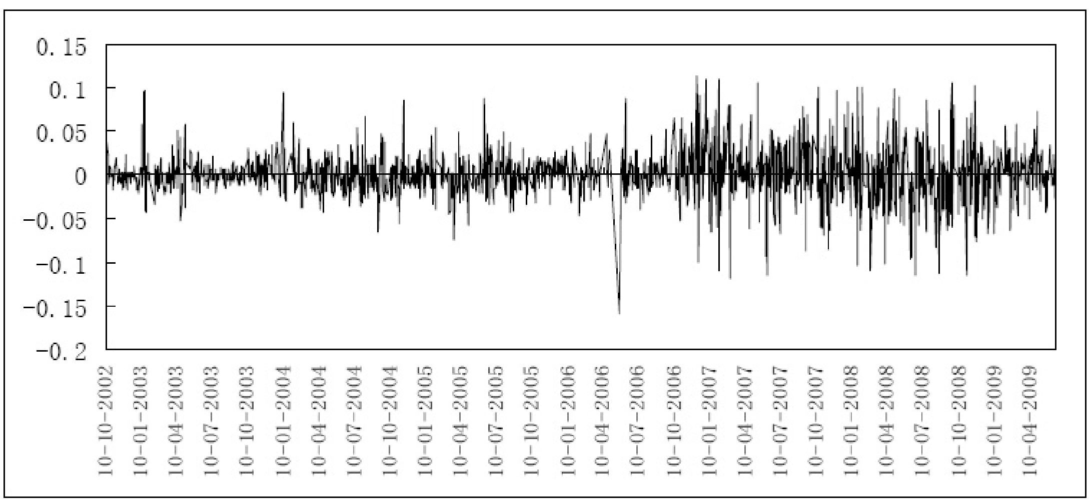

Figure 1: Daily equity log-returns: CUCN

Because one cannot observe a time to default series, we have to find an alternative way to estimate the parameters for both marginal distribution and copulas. We adopt the IFM method

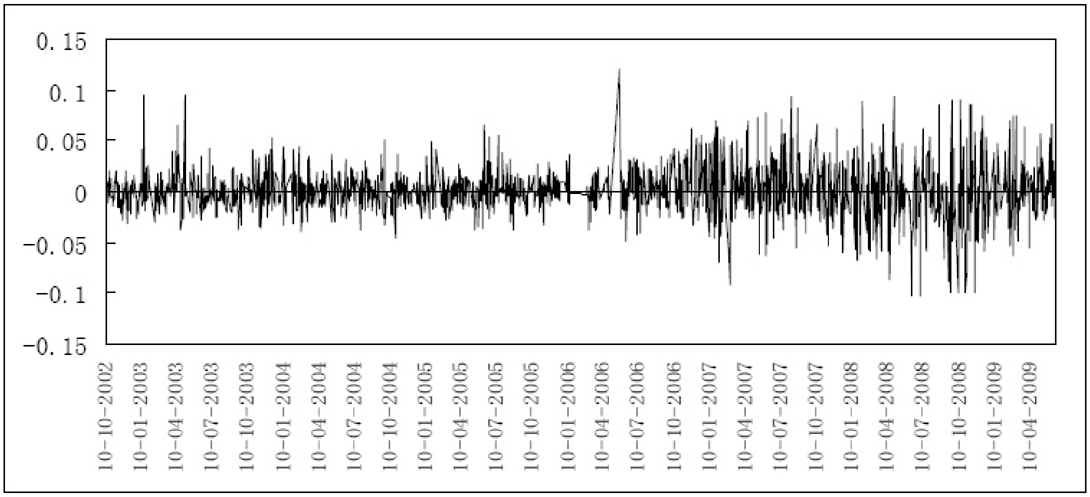

Figure 2: Daily equity log-returns: CMB

by using the corresponding equity prices for each obligation in CDO which is introduced in Cherubini and Luciano (2004).

We choose six stocks from different typical industries to describe the dependence structure in the CDO. The sample data set contains 1578 daily observations from October 9, 2002 to June 9, 2009. We report the log-returns of two equities (CUCN and CMB) in Figure 1 and Figure 2. The other four equities have similar behaviors.

4.2 Model and Parameter Estimation

As many researches point out, the returns pattern of financial data always shows heteroscedasticity behavior. Our figures also confirmed this. So we decide to use the $GARCH(1,1)-t$ model for each log-return series.

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha + \beta y_{t-1}^2 + \gamma \sigma_{t-1}^2 \end{aligned}$$

Here $\varepsilon_t \sim i.i.d.t_\upsilon(0,1)$. The estimates of the parameters of this model are reported in Table 2.

As we can see in Table 2, all GARCH parameters are statistically significant.

Table 2: Estimates of GARCH(1,1) parameters and standard errors

	CUCN	CMB	WISCO	Sinopec	DZT	MINLIST
α	0.00000586	0.00000399	0.0000128	0.00000507	0.00000378	0.0000033
(Std err.)	(0.00000281)	(0.00000176)	(0.00000427)	(0.00000216)	(0.0000017)	(0.0000012)
β	0.086399	0.055111	0.220725	0.085423	0.097305	0.066558
(Std err.)	(0.016883)	(0.011630)	(0.032421)	(0.015548)	(0.017803)	(0.012678)
γ	0.918322	0.940697	0.807365	0.91312	0.906847	0.935021
(Std err.)	(0.013624)	(0.011867)	(0.020844)	(0.014519)	(0.013938)	(0.011925)
υ	3.953288	5.928757	4.459542	4.999693	5.508508	6.117094
(Std err.)	(0.462773)	(0.819326)	(0.360773)	(0.714801)	(0.741022)	(0.648769)

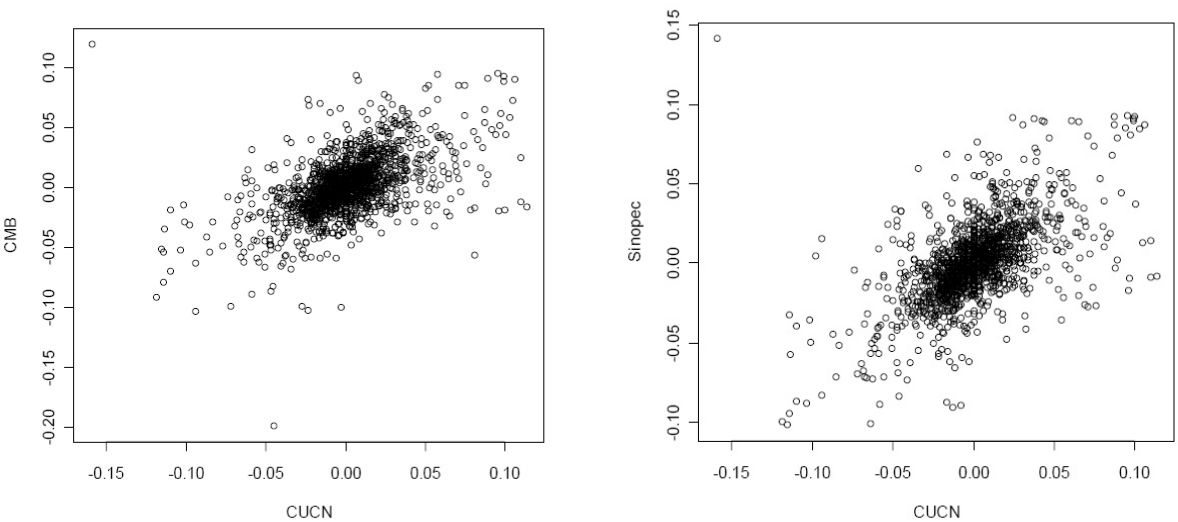

Figure 3: Scatter plot of the dependence between CUCN and CMB/CUCN and Sinopec

To obtain joint default distribution, we consider five copulas, Gaussian copula, Student's t, Gumbel, Frank and Clayton copula. Gaussian copula is the most widely used copula in past years. When we adopt Gaussian copula, we need to calculate the variance and-covariance matrix which contains $\frac{n\times(n-1)}{2}$(n is the number copula's dimension) elements. We report the scatter plot of the dependence between some reference entities in figure 3 and figure 4, where we can see there exist more dispersion in the tails. For such a reason we take Student's t copula into our consideration.

As for the parameters of copula, we firstly adjust the best copula for the standard residual series over the whole sample using Akaike information criterion (AIC) which is computed by the following formula.

$$AIC = -2L(\hat{\theta}; x) + 2q \tag{14}$$

Where q is the number of parameters of copula and $L(\hat{\theta}; x)$ is the maximized log-likelihood value. The copula having the smallest AIC value is the best one.We show the results in Table 3. Obviously Student's t copula is the best one. We, then, use GOF test which we briefly introduced before to test whether the best copula is stable. The p-value we get is equal to 0.003, hence the null hypothesis is rejected and the data set does not remain static.

Table 3: Copula fitting results.

Copula	Parameters	AIC	Convergence
Gaussian	0.895(0.0037); 0.891(0.0039) 0.907(0.0033); 0.878(0.0044) 0.879(0.0043); 0.881(0.0042) 0.895(0.0037); 0.861(0.0049) 0.862(0.0049); 0.887(0.004) 0.866(0.0048); 0.869(0.0046) 0.867(0.0047); 0.867(0.0047) 0.861(0.0049)	-10879.8	T
Student's t	0.918(0.0035); 0.924(0.0033) 0.929(0.0031); 0.911(0.0039) 0.909(0.0039); 0.91(0.0038) 0.917(0.0036); 0.894(0.0045) 0.892(0.0046); 0.916(0.0036) 0.901(0.0042); 0.903(0.0042) 0.899(0.0043); 0.896(0.0044) 0.898(0.0043); 6.29(0.3097)	-11857.2	T
Gumbel	3.39(0.0306)	-10640.8	T
Frank	11.047(0.1116)	-11715	T
Clayton	1.392(0.0197)	-5515.56	T

Figures in the brackets are stand errors. For Student's t copula, the last parameter is the degree of freedom and "T" presents "True".

4.3 Copula Change Analysis

In the following step, we use binary segmentation procedure described in Section 2.3 to detect the change of copulas. We check each sub samples carefully and find out that in most time period, Frank copula and Student's t copula fit better to the data, and in a short time period, Gaussian copula domain the dependent structure in data matrix. We compute a relative AIC value to compare the result of each copula in fitting the data. We choose Student's t copula to be a benchmark and calculate the relative value by $[AIC(t-copula) - AIC(copula)/n.obs.]$ (Here $n.obs.$ is the number of observations). The relative result is reported in Figure 5 (with 16 sub samples). By comparing the relative result in different segments, we can obtain the change time for copula and the results are reported in Table 4.

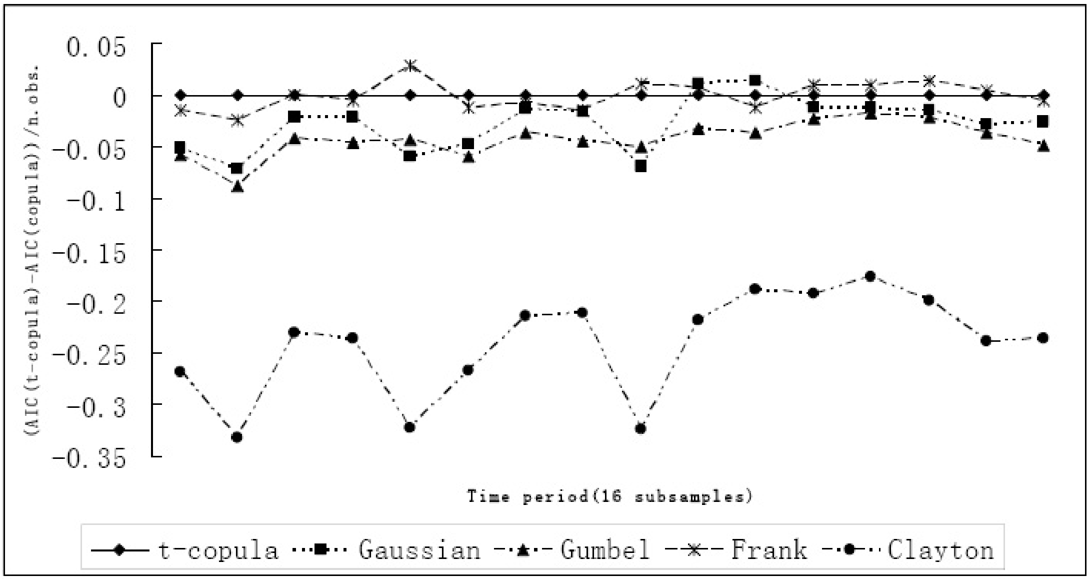

Figure 4: The relative AIC of copulas

As can be seen, the duration of CDO is divided into 9 stages denoted by I, II, III, IV, V, VI, VII, VIII and IX. Each stage has a different copula function that fit best for the data. Some of the changes points are coincide with real financial incidents.

In Nov 7. 2005, copula family changes from Student's t copula to Frank copula. This change corresponds to equity division reform in Chinese stock market. Under this reform, the stock market in China ended its 4 years bearish market and turned into bull market.

On May 24. 2007, copula family changes from Gaussian copula to Frank copula. This date corresponds to the subprime mortgage crisis in American. Under the impact of this crisis, Chinese stock market becomes more and more fluctuant. Before this date, the standard deviation of the Shanghai Composite Index is 593.9959, while after this date it becomes 1242.5.

Table 4: Changes of copula.

Stage	Time Period	Copula	Parameters	Change Time
I	10/10/2002-05/08/2003	Student's t	C_1	
II	06/08/2003-30/12/2003	Frank	11.106(0.4527)	06/08/2003
III	31/12/2003-06/06/2004	Student's t	C_2	31/12/2003
IV	07/06/2004-28/10/2004	Frank	11.606(0.4596)	07/06/2004
V	29/10/2004-06/11/2005	Student's t	C_3	29/10/2004
VI	07/11/2005-23/07/2006	Frank	10.845(0.3551)	07/11/2005
VII	24/07/2006-23/05/2007	Gaussian	C_4	24/07/2006
VIII	24/05/2007-08/01/2009	Frank	11.216(0.2282)	24/05/2007
IX	09/01/2009-10/07/2009	Student's t	C_5	09/01/2009

Figures in the brackets are stand errors. In the first stage, C_1= (0.957, 0.964, 0.965, 0.934, 0.92, 0.935, 0.955, 0.926, 0.911, 0.949, 0.923, 0.911, 0.931, 0.908, 0.903), corresponding stand errors are (0.0056, 0.0047, 0.0044, 0.0087, 0.01, 0.008, 0.0057, 0.0094, 0.0112, 0.0063, 0.0096, 0.0115, 0.0088, 0.0116, 0.0121). The degree of freedom is 4.419 with stand error 0.4927. In the third stage, C_2= (0.94, 0.905, 0.912, 0.921, 0.919, 0.912, 0.917, 0.936, 0.921, 0.898, 0.886, 0.918, 0.863, 0.894, 0.898), corresponding stand errors are (0.0103, 0.0164, 0.0147, 0.0137, 0.0147, 0.0152, 0.0141, 0.0111, 0.014, 0.017, 0.0191, 0.0148, 0.0221, 0.0181, 0.0175). The degree of freedom is 6.279 with stand error 1.3412. In the fifth stage, C_3= (0.933, 0.928, 0.934, 0.92, 0.916, 0.929, 0.927, 0.914, 0.912, 0.922, 0.905, 0.905, 0.911, 0.901, 0.915), and stand errors are (0.0075, 0.0078, 0.007, 0.0087, 0.009, 0.0076, 0.0078, 0.0095, 0.0096, 0.0084, 0.0101, 0.01022, 0.0094, 0.0104, 0.0091). The degree of freedom is 6.271 and stand error is 0.8312. In the seventh stage, C_4= (0.868, 0.874, 0.84, 0.828, 0.828, 0.864, 0.85, 0.832, 0.85, 0.856, 0.846, 0.846, 0.796, 0.82, 0.844) with stand errors (0.0129, 0.0126, 0.0157, 0.0168, 0.017, 0.0135, 0.0146, 0.0163, 0.0148, 0.0143, 0.0153, 0.0155, 0.0197, 0.0178, 0.0154). In the ninth stage, C_5= (0.923, 0.921, 0.927, 0.917, 0.917, 0.914, 0.936, 0.903, 0.893, 0.897, 0.889, 0.909, 0.91, 0.888, 0.899) and stand errors is (0.0133, 0.013, 0.0123, 0.0143, 0.0144, 0.0147, 0.0111, 0.0168, 0.0184, 0.0169, 0.0189, 0.0155, 0.0154, 0.0192, 0.0181). The degree of freedom is 6.415 and stand error is 1.2896.

4.4 CDO Pricing

In Jan 9. 2009, copula family changes from Frank copula to Student's t copula. This change point corresponds to the relatively stability of stock market after the financial crisis. In China the stock index started to rise slowly.

After we got the dynamic dependent structure of each CDO items we can simulate its default time and then calculate its fair spreads. By using the method introduced in Part 3, we simulate every firm's default time under each stage. In order to obtain the number of defaults that may occur prior to maturity date, we firstly generate 10000 random variates from the copula that best fit for the data sets, and then using equation (3.1) and (3.2) .we can calculate the corresponding default time . Figure 6 reports the histograms of the distribution of the simulated time to default of CUCN and CMB.

For the given recovery rates and simulated default numbers, we calculated the expected loss and fair spread of each tranche for all nine stages. The results are reported in Table 5 and Table 6.

Table 5: Expected loss of each tranche in nine stages.× (1000).

Tranche	Stages and Corresponding Best Copula								
	I	II	III	IV	V	VI	VII	VIII	IX
	t	Frank	t	Frank	t	Frank	Gaussian	t	Frank
A	284.4	449.4	273	461.4	294.9	417.6	256.8	416.4	291.6
B	220.8	280.8	252	241.2	264	230.4	255.6	223.2	283.2
C	150.6	65.4	195	46.5	186	55.5	211.5	50.7	220.8
D	136.2	6	123	5.7	130.5	6.9	75.3	5.1	117.6

Table 6: Fair spread of each tranche in nine stages.

Tranche	Stages and Corresponding Best Copula								
	I	II	III	IV	V	VI	VII	VIII	IX
	t	Frank	t	Frank	t	Frank	Gaussian	t	Frank
A	2.64%	7.95%	3.42%	8.14%	3.26%	7.35%	4.49%	7.31%	3.87%
B	1.89%	2.38%	2.16%	2.04%	2.25%	1.95%	2.20%	1.88%	2.41%
C	1.37%	0.31%	1.32%	0.22%	1.42%	0.26%	1.02%	0.24%	1.40%
D	0.35%	0.02%	0.32%	0.01%	0.33%	0.02%	0.19%	0.01%	0.30%

From Table 5 and Table 6, we detect that the expected loss and fair spread are not the same under different stages. Generally speaking, in stages that Frank copula domains the

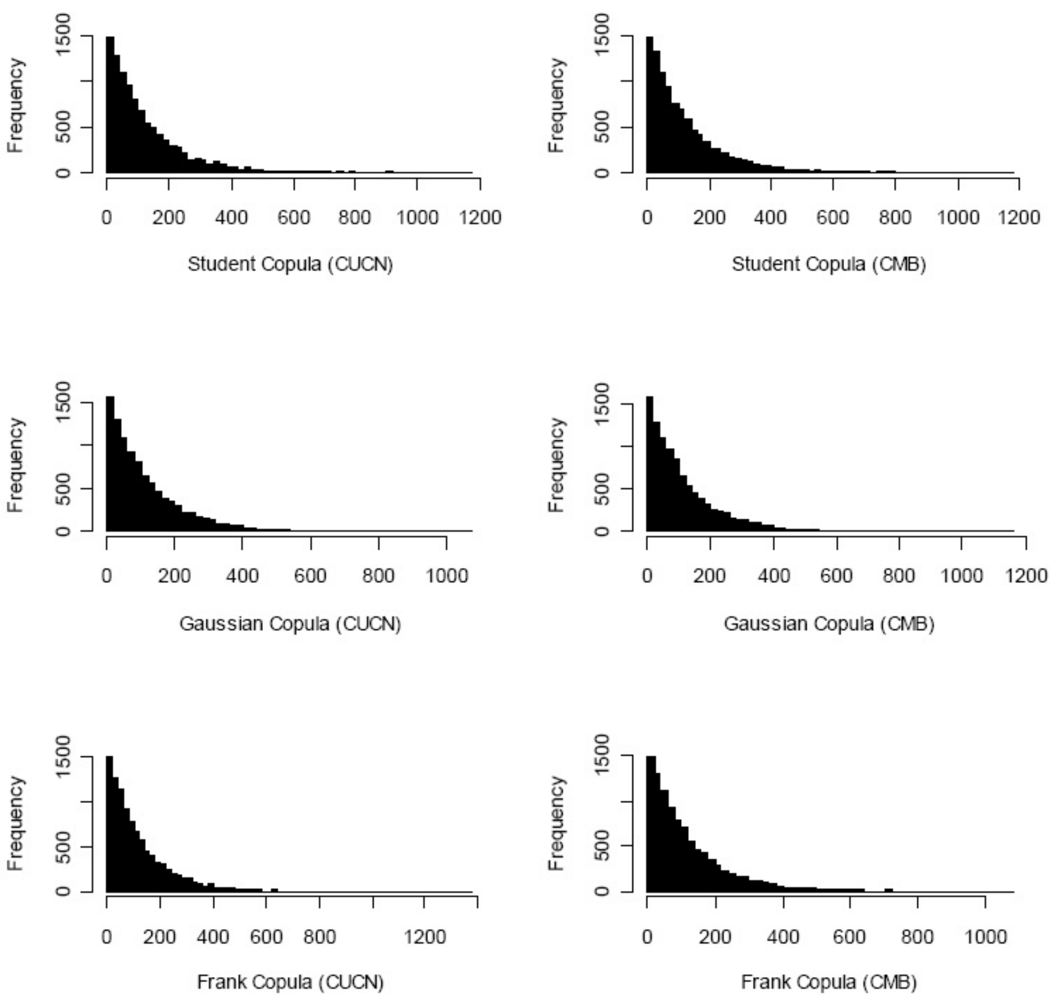

Figure 5: Histogram of simulated default times for CUCN and CMB

dependent structure, the fair spread is higher in tranche A,B and lower fair spread in tranche C,D. More specifically, in stages that Frank copula domains the dependent structure, the fair spread of tranche A is above of 7%, while in stages that Student's t copula domains the dependent structure, the corresponding fair spread is less than 4%. Fair spreads of tranche D in stages with Frank copula are less than 0.1% while for Student's t copula they are all above 0.3%. Expected loss and fair spread under Gaussian copula are close to those under Student's t copula.

4.5 Comparison to static Gaussian and dynamic Gaussian

In order to give a comparison, we use static copula model and dynamic Gaussian copula model to price the same CDO.

As we see in table 3, student's t copula is the best one fitting to the data, so we use student's t copula and it's corresponding parameters to price the same CDO. We also employ Gaussian copula model to calculate a comparable result. The results are presented in table 7.

Table 7: Expected loss and fair spread of each tranche in static copula model.

Copula model	Tranche	Expected loss	Fair spreads
Student's t copula	A	322.2	4.12%
	B	278.4	2.89%
	C	192.6	1.88%
	D	147	0.46%
Gaussian copula	A	244.4	4.41%
	B	202.4	2.09%
	C	155.6	0.77%
	D	67	0.16%

The results in table 7 show that tranche B, C and D have a greater fair spread than that in dynamic copula model which indicate that the tail dependence is well captured by student's t copula. However, this static student's t copula model tends to create too heavy a tail dependence structure in default correlation and lead to misprice to CDO. In static Gaussian copula model, tranches B, C and D have smaller fair spreads than that in static student's t copula which indicates that this model is lack of tail dependence.

In the following part, we choose Gaussian copula as the basic copula and let the parameter change to see the pricing of CDO in dynamic Gaussian copula. We calculate the parameters in nine stages which we divide in dynamic copula model and the results are reported in table 8. Table 9 and table 10 show the expected loss and fair spread respectively in each tranches of nine stages. In order to give an intuitional comparison, we plot the fair spread of each tranche calculated by three models together and report them in figure 7 to figure 10.

Table 8: Estimated results for dynamic Gaussian copula.

Stages	I	II	III	IV	V	VI	VII	VIII	IX
	0.931	0.917	0.927	0.905	0.909	0.868	0.871	0.870	0.901
	0.923	0.916	0.880	0.782	0.903	0.891	0.877	0.892	0.911
	0.942	0.946	0.907	0.907	0.914	0.881	0.854	0.908	0.914
	0.879	0.894	0.901	0.891	0.897	0.844	0.839	0.887	0.882
	0.875	0.889	0.876	0.857	0.886	0.893	0.833	0.890	0.905
	0.893	0.888	0.894	0.795	0.916	0.867	0.870	0.874	0.897
	0.931	0.900	0.909	0.915	0.924	0.857	0.855	0.879	0.914
Gaussian copula	0.869	0.850	0.927	0.888	0.888	0.814	0.831	0.857	0.848
	0.857	0.842	0.886	0.861	0.885	0.849	0.854	0.856	0.865
	0.918	0.897	0.882	0.776	0.907	0.884	0.886	0.886	0.881
	0.871	0.850	0.859	0.790	0.884	0.857	0.848	0.893	0.845
	0.853	0.860	0.876	0.808	0.888	0.833	0.859	0.890	0.898
	0.891	0.870	0.844	0.857	0.890	0.828	0.804	0.898	0.871
	0.868	0.854	0.857	0.866	0.877	0.862	0.819	0.891	0.862
	0.872	0.829	0.858	0.885	0.894	0.770	0.848	0.880	0.852

Table 9: Expected loss of each tranche in nine stages.×(1000).

Tranche	Stages and Corresponding Best Copula								
	I	II	III	IV	V	VI	VII	VIII	IX
A	340.8	290.4	352.8	318	289.2	309.6	256.8	297.6	325.2
B	328.2	257.7	329.1	276.9	267.6	250.2	255.6	257.7	304.2
C	244.2	222	245.4	268.2	234.6	261	211.5	229.2	240.6
D	130.2	104.7	126.3	101.1	109.2	93.6	75.3	100.5	118.8

Table 10: Fair spread of each tranche in nine stages.

Tranche	Stages and Corresponding Best Copula								
	I	II	III	IV	V	VI	VII	VIII	IX
A	5.27%	4.73%	5.30%	5.76%	4.99%	5.58%	4.49%	4.88%	5.16%
B	3.56%	3.01%	3.67%	3.31%	2.99%	3.21%	2.20%	3.08%	3.39%
C	1.92%	1.51%	1.93%	1.62%	1.56%	1.46%	1.02%	1.50%	1.78%
D	0.40%	0.32%	0.39%	0.31%	0.34%	0.29%	0.19%	0.31%	0.37%

In the last four figures, we can see that dynamic copula model has the greatest range in fair spread. Dynamic Gaussian copula model shows a smaller range but the wave direction of the

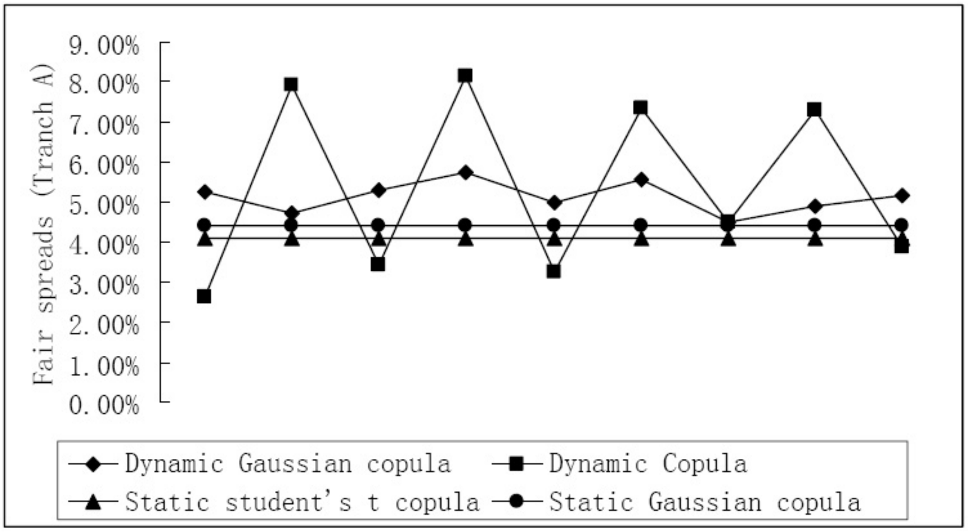

Figure 6: Fair spread of tranche A by three models

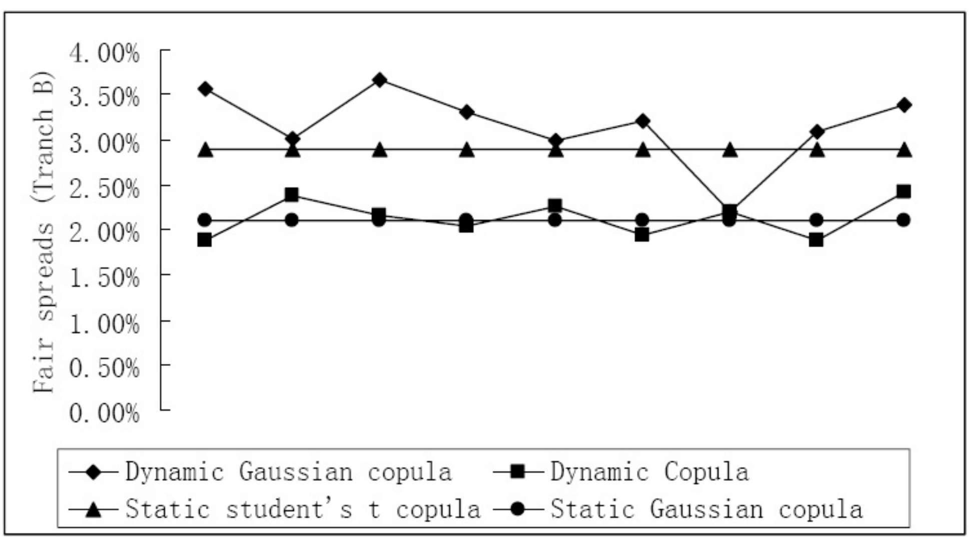

Figure 7: Fair spread of tranche A by three models

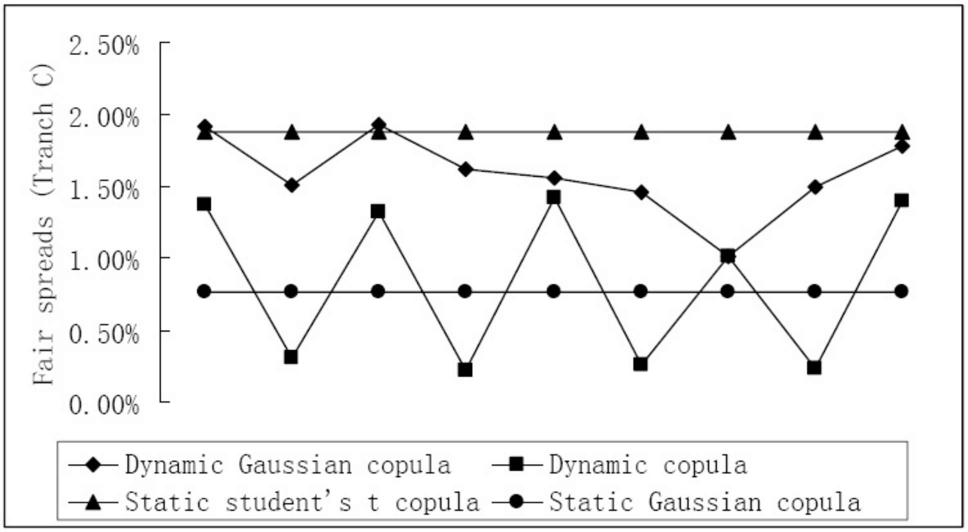

Figure 8: Fair spread of tranche A by three models

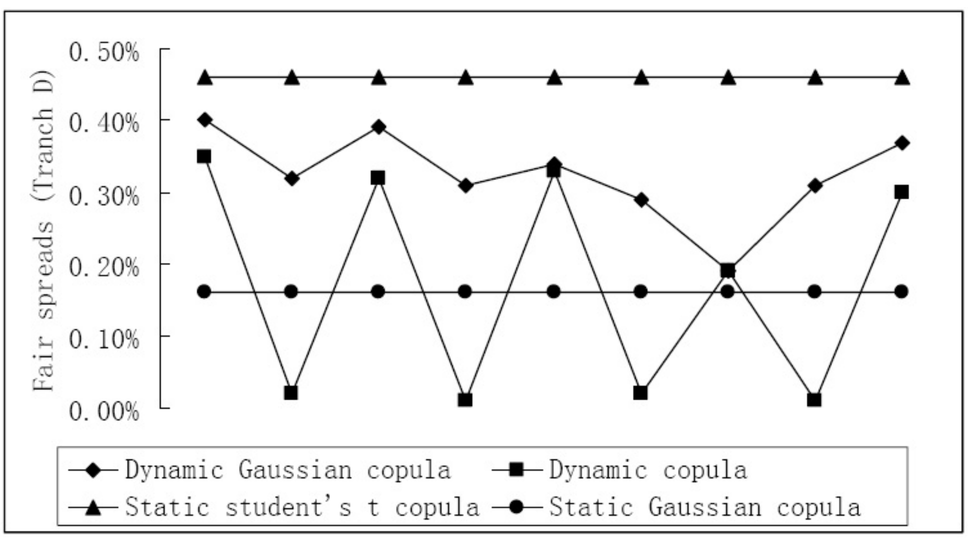

Figure 9: Fair spread of tranche A by three models

results is similar with dynamic copula model. In the figure 9 and figure 10, which is in case of tranche C and D, the fair spread calculated by static student's t copula model is remarkably higher than that by dynamic copula model and dynamic Gaussian copula model. As student's t copula is tend to stress the tail dependence, the higher fair spread in tranche C and D illustrate a clustering phenomenon in dependence structure, which is approved in Xianhua Peng and Steven Kou(2009). However, the static student's t copula model suggest that the default correlation is exist all the time and remain static, which is not proper because the dependence structure tend to change instead of static even in case of tail dependence.

5 Conclusion

In this paper, we use dynamic copula method to describe the dynamic default correlation between the obligors of multi–name credit derivatives. We choose GOF test to check the stability of a single copula model and find the static copula model is not reliable when dealing with dependent structure of financial series. Based on this fact, the binary segmentation procedure is used to detect the change point of default correlation during credit derivative's life. The empirical results suggest that there are nine different stages in the whole sample and each stage has a best copula to fit the data set. In each stage we calculate the expected loss and fair spread for every tranche Results show that spreads of a CDO are not always the same. Under different situation, even the same CDO has different risk and hence has different expected loss and fair spread. This explains why investors suffered so much loss when financial crisis happened. We also give a comparison among static copula model, dynamic Gaussian copula model and dynamic copula model. The comparison suggest that static student's t copula model tend to stress the tail dependence in default correlation and dynamic Gaussian copula model has similar results as dynamic model but the result is less sensitive to the change of dependence structure.

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